Computation, Machines and Formal Languages - Exam (spring 2002) Moed A

Lecturer: Prof. Michael Ben-Or.

Time: 2.5 hours.

Write your answers on the exam sheet.

If the answer is yes/no, **write down the answer that you chose**.

Answer all the questions.

Unless stated otherwise, you have to add a short explanation (within the designated area for each question) for your answers. Answers that will not be accompanied by explanations will not be credited. The explanation should include the main steps in the proof, do not get into technical details, notations etc. If your explanation is given by a contradicting example, you have to state the example together with a short explanation of the contradiction. If you have an example that contradicts an unproven but a reasonable assumption (such as $P \neq NP$), state the example, the reasonable assumption, and a short explanation of the contradiction.

Definitions and Notations: (identical to those we used in class)

For a language $\mathcal{L}$, we denote by $\overline{\mathcal{L}}$ the language $\Sigma^* \setminus \mathcal{L}$.

Complexity Classes:
REG is the class of regular languages.
CFL is the class of context-free languages.
R is the class of languages that are decidable by Turing Machines (TM).

RE is the class of languages that are recognizable by TM's.
P is the class of languages that are decidable by deterministic TM's in polynomial time.
NP is the class of languages that are decidable by non-deterministic TM's in polynomial time.
$PSPACE$ is the class of languages that are decidable by deterministic TM's in polynomial space.
L is the class of languages that are decidable by deterministic TM's in logarithmic space.
NL is the class of languages that are decidable by non-deterministic TM's in logarithmic space.
For a class of languages $\mathcal{C}$, we denote by $co\mathcal{C}$ the class of languages that their complements are in $\mathcal{C}$.

Computational problems:
$SUBSET-SUM = \{(S,t)$: S is a (multi) set of numbers that contains a subset that sums to $t\}$
$HAM-PATH = \{(G,s,t)$: G is a directed graph, that contains a path from s to t, that visits every node in the graph exactly once$\}$
We say that a Boolean formula is $3-cnf$ if it is of the form:

$$\bigwedge_i (a_1^i \vee a_2^i \vee a_3^i)$$

where a_j^i is a variable or its negation.
$3-SAT = \{\phi : \phi$ is a satisfiable $3-cnf\}$
$k-color = \{G$: given k colors, it is possible to assign a color to each node in the graph G, such that no two adjacent nodes get the same color$\}$
$ALL_{CFG} = \{G : G$ is a context free grammar, and $L(G) = \Sigma^*\}$
$ALL_{DFA} = \{A : A$ is a deterministic finite automaton, and $L(A) = \Sigma^*\}$
$QBF = \{\phi : \phi$ is a true fully quantified Boolean formula$\}$

Reductions:
For two languages $\mathcal{L}_1, \mathcal{L}_2$, $\mathcal{L}_1 \leq_P \mathcal{L}_2$ denotes that there is a polynomial-time mapping reduction from $\mathcal{L}_1$ to $\mathcal{L}_2$.
For two languages $\mathcal{L}_1, \mathcal{L}_2$, $\mathcal{L}_1 \leq_L \mathcal{L}_2$ denotes that there is a logarithmic-space mapping reduction from $\mathcal{L}_1$ to $\mathcal{L}_2$.

1. (12 points) Draw a **minimal deterministic** finite automaton for the language $\mathcal{L} = \{w : |w| = 2k, k \geq 0\}$ (over $\Sigma = \{0, 1\}$).
   Give a short proof for the correctness and the minimality.

2. (13 points) Let us define the following languages over $\Sigma = \{a, b\}$:

$$L = \{a^n b^m a^k : n \geq m \geq k\}$$

$$K = a^* b^*$$

For each one of the following languages, write the smallest class of languages that contains it out of the following list: REG, CFL, or P.

You do not have to prove or explain your answers.

(a) $L \cap K$

(b) $\overline{L}$

(c) $\overline{L} \cap K$

(d) L

3. (13 points)

(a) For each one of the following languages, write the smallest class that contains it out of the following list: R, RE, $coRE$, or none of these classes.

i. ALL_{CFG}

ii. ALL_{DFA}

iii. $L = \{< M >: L(M)$ contains more than 100 words$\}$

(b) For each one of the languages for which you answered R, write the smallest class that contains it out of the following list: L, NL, P.

4. (12 points) For each one of the following functions, answer whether it is $\Theta(1)$, $\Theta(\log n)$, or $\Theta(n)$.

   (a) $f_1(n) = max\{|K(x) - K(xx^R)| : x \in \{0,1\}^n\}$

(b) $f_2(n) = max\{|K(x) - K(y)| : x, y \in \{0,1\}^{2n}$, and in each one of the strings x and y, exactly half the symbols are 1 and half are $0\}$

5. (13 points) Let $\mathcal{L}$ be the following language. $\mathcal{L} = \{\langle \varphi_1, \varphi_2 \rangle$: exactly one of φ_1, φ_2 is satisfiable $\}$. Answer the following questions.

   (a) Is $\mathcal{L} \in NP$?

(b) Is $\mathcal{L} \in coNP$?

(c) Is $\mathcal{L} \in PSPACE$?

6. (13 points) Under the assumption that $3 - color \notin coNP$, are the following true?

   (a) $3 - SAT \in coNP$?

   (b) $\overline{3 - SAT} \notin P$?

   (c) $QBF \in NP$?

7. (12 points) Let Σ be some fixed alphabet. The operation of *shuffle* is important in the study of concurrent systems. If $x,\ y \in \Sigma^*$, then $x \,||\, y$ is the set of all strings that can be made by shuffling x and y together like a deck of cards, that is, mixing the letters of x with the letters of y, while keeping the order of their appearance in the original words. For example,

$$ab \,||\, cd = \{abcd, acbd, acdb, cabd, cadb, cdab\}$$

The shuffle of two languages A and B, denoted $A \,||\, B$, is the set of strings obtained by shuffling a string from A with a string from B:

$$A \,||\, B = \bigcup_{\substack{x \in A \\ y \in B}} x \,||\, y$$

A class $\mathcal{C}$ of languages is **closed** under shuffle if for all $A, B \in \mathcal{C}$, we have $A \,||\, B \in \mathcal{C}$. Are the following classes closed under shuffle?

(a) REG

(b) P

(c) NP

8. (12 points) A language C **separates** two disjoint languages A, B if $A \subseteq C$ and $B \subset \bar{C}$ (see Figure 1). Prove that for all pairs of disjoint languages A, B in $coRE$ there is a language in R that separates them.

   Hint: All languages in RE have enumerators that print them.

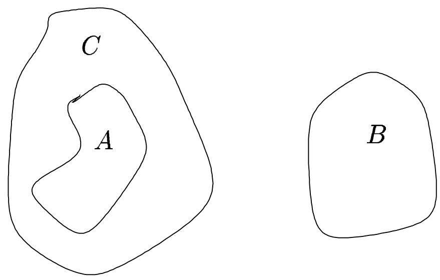

Figure 1: A separating language