CHAPTER 15

PERTURBATION THEORY OF TWO-DIMENSIONAL REAL AUTONOMOUS SYSTEMS

1. Two-dimensional Linear Systems

Consider the real linear system

$$\begin{aligned} x_1' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2 \end{aligned} \qquad \left(' = \frac{d}{dt}\right) \tag{1.1}$$

where a, b, c, d are real constants such that the determinant $ad - bc$ does not vanish. Clearly $(x_1,x_2) = (0,0)$ is then the *only* critical point of this system, that is, the only point where the right member of (1.1) vanishes. Let the coefficient matrix of (1.1) be denoted by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then (1.1) can be written as $x' = Ax$, where $x = (x_1,x_2)$. Let A have the characteristic roots λ, μ. These roots can be real or complex, but if one is complex, say $\lambda = \alpha + i\beta$ (α, β real, $\beta \neq 0$), then $\mu = \alpha - i\beta$ is the other root, for the coefficients of the characteristic equation for A are real. It is known that there exists a *real* nonsingular constant matrix T such that, if $y = Tx$, then the transformed system $y' = (TAT^{-1})y$ has a real coefficient matrix $J = (TAT^{-1})$ which has one of the following real canonical forms:

(I) $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ $(\lambda \neq 0)$ (II) $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ $(\mu < \lambda < 0$, or $0 < \mu < \lambda)$

(III) $\begin{pmatrix} \lambda & 0 \\ \gamma & \lambda \end{pmatrix}$ $(\lambda \neq 0, \gamma > 0)$ (IV) $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ $(\lambda < 0 < \mu)$

(V) $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ $(\alpha \neq 0, \beta \neq 0)$ (VI) $\begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$ $(\beta \neq 0)$

Thus, in order to discuss the nature of the orbits of (1.1) near (0,0), one may assume A has one of the forms (I) through (VI).

Before taking up each of these cases individually, a matter of notation will be settled. In general, a solution of a two-dimensional system

$$x_1' = g_1(x_1,x_2) \qquad x_2' = g_2(x_1,x_2) \tag{1.2}$$

will be denoted by $\varphi = (\varphi_1,\varphi_2)$, and it will often be convenient to consider the *polar functions* ρ, ω, associated with the solution φ, defined by

$$\rho(t) = (\varphi_1^2(t) + \varphi_2^2(t))^{\frac{1}{2}} \qquad \omega(t) = \tan^{-1}\frac{\varphi_2(t)}{\varphi_1(t)}$$

It is stressed that ρ, ω are defined with respect to a *particular solution* φ

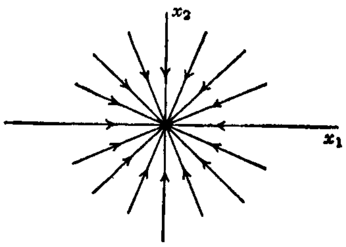

FIG. 3. (I) Proper node, $\lambda < 0$.

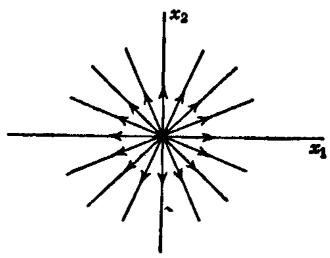

FIG. 4. (I) Proper node, $\lambda > 0$.

of (1.2), and are consequently *functions of t*. Thus ρ, ω are to be distinguished from the *polar coordinates* r,θ in the (x_1,x_2) plane defined by

$$r = (x_1^2 + x_2^2)^{\frac{1}{2}} \qquad \theta = \tan^{-1}\frac{x_2}{x_1}$$

just as the solution coordinate functions φ_1,φ_2 are to be distinguished from the Cartesian coordinates x_1,x_2 in the plane.

(I) Here the system is given by

$$x_1' = \lambda x_1 \qquad x_2' = \lambda x_2$$

and therefore, if (c_1,c_2) is any initial point except (0,0), a solution through this point is given by $\varphi_1(t) = c_1e^{\lambda t}$, $\varphi_2(t) = c_2e^{\lambda t}$. If $\lambda < 0$, then $\rho(t) \to 0$ as $t \to +\infty$, and if $\lambda > 0$, $\rho(t) \to 0$ as $t \to -\infty$. The orbit through (c_1,c_2) is an open half line passing through this point and with an end point at (0,0). See Figs. 3 and 4, where the arrows indicate the direction of increasing t. This type of critical point is called a *proper node*. Its distinguishing feature is that *every orbit tends to the origin in a definite direction* as $t \to +\infty$ (for $\lambda < 0$), or as $t \to -\infty$ (for $\lambda > 0$), and, *given any direction, there exists an orbit which tends to the origin in this direction.* Thus the origin is (asymptotically) stable in case $\lambda < 0$, and unstable when $\lambda > 0$.

(II) The system for Case (II) is

$$x_1' = \lambda x_1 \qquad x_2' = \mu x_2$$

and the solution passing through $(c_1,c_2) \neq (0,0)$ at $t = 0$ is given by $\varphi_1(t) = c_1e^{\lambda t}$, $\varphi_2(t) = c_2e^{\mu t}$. Assume $\mu < \lambda < 0$, for example. Then as $t \to +\infty$, $(\varphi_1(t),\varphi_2(t)) \to (0,0)$, and if $c_1 \neq 0$, $\varphi_2(t)/\varphi_1(t) = (c_2/c_1)e^{(\mu-\lambda)t} \to 0$, as $t \to +\infty$. If $c_1 = 0$, $c_2 \neq 0$, $(\varphi_1(t),\varphi_2(t)) = (0,c_2e^{\mu t})$, which is just the open positive or negative x_2 axis, according as $c_2 > 0$ or $c_2 < 0$. In this

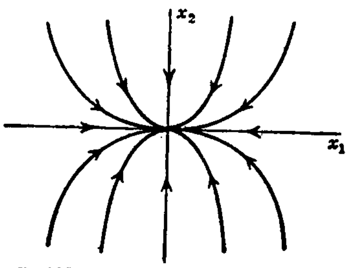

FIG. 5. (II) Improper node, $\mu < \lambda < 0$.

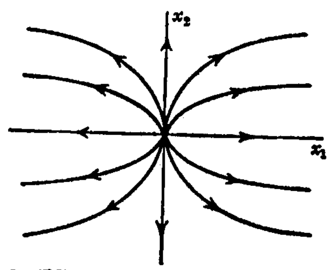

FIG. 6. (II) Improper node, $0 < \mu < \lambda$.

case, the origin is called an *improper node*. A qualitative picture of the orbits is shown in Figs. 5 and 6. Here, *every orbit, except one, has the same limiting direction at the origin*. The origin is (asymptotically) stable in case $\mu < \lambda < 0$, and unstable when $0 < \mu < \lambda$.

(III) The equations in this case are

$$x_1' = \lambda x_1 \qquad x_2' = \gamma x_1 + \lambda x_2$$

and it is easy to see that $\varphi_1(t) = c_1e^{\lambda t}$, $\varphi_2(t) = (c_2 + c_1\gamma t)e^{\lambda t}$, is the solution passing through (c_1,c_2) at $t = 0$. Suppose $\lambda < 0$, for example. Then as $t \to +\infty$, φ_1 and φ_2 tend to 0. If $c_1 \neq 0$, $\varphi_2(t)/\varphi_1(t) = c_2/c_1 + \gamma t \to \pm\infty$,

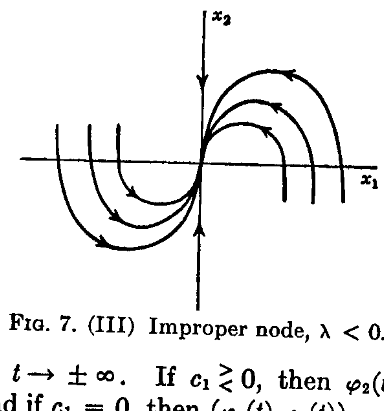

FIG. 7. (III) Improper node, $\lambda < 0$.

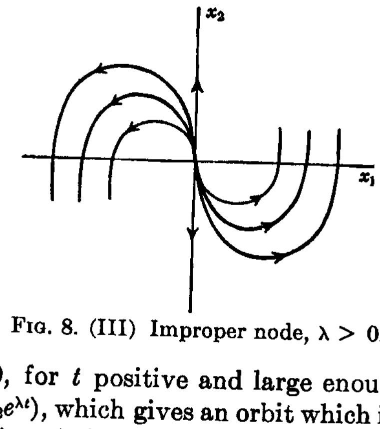

FIG. 8. (III) Improper node, $\lambda > 0$.

as $t \to \pm\infty$. If $c_1 \gtrless 0$, then $\varphi_2(t) \gtrless 0$, for t positive and large enough, and if $c_1 = 0$, then $(\varphi_1(t),\varphi_2(t)) = (0,c_2e^{\lambda t})$, which gives an orbit which is a half x_2 axis. Also, if $c_1 \neq 0$, $\varphi_2'(t)/\varphi_1'(t) = (\gamma/\lambda) + \varphi_2(t)/\varphi_1(t) \to \pm\infty$, as $t \to \pm\infty$. Thus *every orbit has the same limiting direction at* (0,0). The origin in this case is also called an *improper node*. The nature of the orbits is sketched in Figs. 7 and 8.

(IV) Here the equations are

$$x_1' = \lambda x_1 \qquad x_2' = \mu x_2$$

and a solution is given by $\varphi_1(t) = c_1 e^{\lambda t}$, $\varphi_2(t) = c_2 e^{\mu t}$, where now $\lambda < 0$, $\mu > 0$. If $|\lambda| = |\mu|$, the orbits would be rectangular hyperbolas. In the general case, the orbits resemble these hyperbolas; see Fig. 9. Here, if $(c_1,c_2) \neq (0,0)$, $\varphi_1(t) \to 0$, $\varphi_2(t) \to \pm\infty$, according as $c_2 > 0$ or $c_2 < 0$. In this case, the origin is called a *saddle point*.

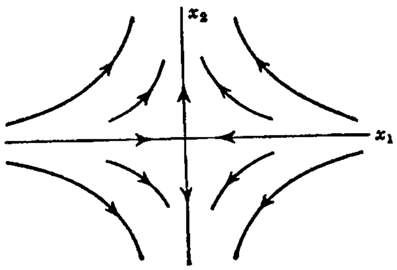

FIG. 9. (IV) Saddle point, $\lambda < 0 < \mu$.

(V) In this case

$$x_1' = \alpha x_1 + \beta x_2$$
$$x_2' = -\beta x_1 + \alpha x_2$$

and the solution which passes through (c_1,c_2) at $t = 0$ is given by

$$\varphi_1(t) = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t) \qquad \varphi_2(t) = e^{\alpha t}(-c_1 \sin \beta t + c_2 \cos \beta t)$$

If $\rho_0^2 = c_1^2 + c_2^2$, this solution may be written $\varphi_1(t) = \rho_0 e^{\alpha t} \cos (\beta t - \delta)$, $\varphi_2(t) = -\rho_0 e^{\alpha t} \sin (\beta t - \delta)$, where $\cos \delta = c_1/\rho_0$ and $\sin \delta = c_2/\rho_0$. The polar functions ρ, ω for this solution are $\rho(t) = \rho_0 e^{\alpha t}$, $\omega(t) = -\beta t + \delta$, and hence $\rho = Ce^{-(\alpha/\beta)\omega}$, where $C = \rho_0 e^{(\alpha/\beta)\delta}$, which is a spiral. Thus the origin in this case is called a *spiral point*. (Alternate terms for such a point are *vortex* and *focus*.) See Figs. 10 and 11.

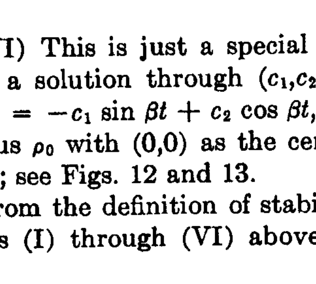

FIG. 10. (V) Spiral point, $\alpha < 0$, $\beta < 0$.

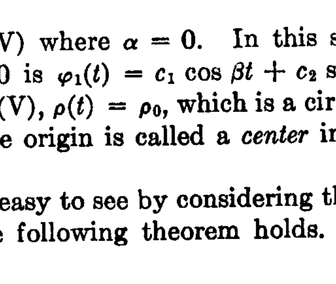

FIG. 11. (V) Spiral point, $\alpha > 0$, $\beta < 0$.

(VI) This is just a special case of (V) where $\alpha = 0$. In this situation a solution through (c_1,c_2) at $t = 0$ is $\varphi_1(t) = c_1 \cos \beta t + c_2 \sin \beta t$, $\varphi_2(t) = -c_1 \sin \beta t + c_2 \cos \beta t$, or, as in (V), $\rho(t) = \rho_0$, which is a circle of radius ρ_0 with (0,0) as the center. The origin is called a *center* in this case; see Figs. 12 and 13.

From the definition of stability, it is easy to see by considering the six cases (I) through (VI) above that the following theorem holds. The pictures in Figs. 3 through 13 give a nice qualitative idea of the notion of stability in each of the cases.

Theorem 1.1. *Necessary and sufficient for the origin to be stable for the system* (1.1) *is that the characteristic roots of the real nonsingular coefficient matrix A should have negative or zero real parts.*

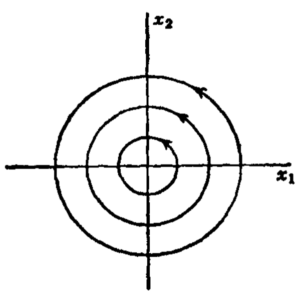

FIG. 12. (VI) Center, $\beta < 0$.

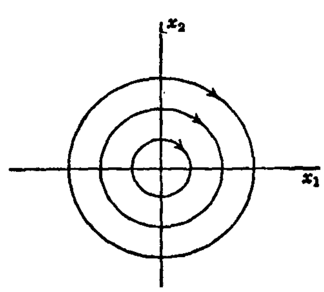

FIG. 13. (VI) Center, $\beta > 0$.

2. Perturbations of Two-dimensional Linear Systems

Consider now the *nonlinear* two-dimensional real autonomous system

$$\text{(NL)} \qquad \begin{aligned} x_1' &= ax_1 + bx_2 + f_1(x_1,x_2) \\ x_2' &= cx_1 + dx_2 + f_2(x_1,x_2) \end{aligned}$$

where a, b, c, d are real constants, $ad - bc \neq 0$, and f_1,f_2 are real continuous functions defined in some circle about the origin $(x_1,x_2) = (0,0)$ with radius $r_0 > 0$. The functions f_1 and f_2 are called *perturbations*, and the system (NL) will be referred to as the *perturbed system* corresponding to the *linear* system

$$\text{(L)} \qquad x_1' = ax_1 + bx_2 \qquad x_2' = cx_1 + dx_2$$

Intuitively, if the perturbations f_1 and f_2 are "small" in some sense, one would expect that the behavior of the orbits of (NL) near the origin in the (x_1,x_2) plane would be very similar to the behavior of the orbits of (L). It will be shown that this is in general true, provided that f_1,f_2 satisfy certain minimum assumptions.

In addition to the given assumptions on f_1,f_2, it will be assumed that

$$f_1 = o(r) \qquad f_2 = o(r) \qquad (\text{as } r \to 0+) \tag{2.1}$$

This guarantees that the perturbations tend to zero faster than the linear terms in (NL). Also, it is easily seen that this condition, and the fact that $ad - bc \neq 0$, imply that the origin is an *isolated* critical point for (NL); that is, there exists a circle about the origin in which the origin is the only point where the right member of (NL) vanishes. An isolated